Mass Energy Equivalence

By the Work-Energy Theorem:

$$W = KE$$

Where W is work done,KE is kinetic energy,
In this case, we will consider $PE = 0$
Where PE is potential energy
so:

$$W = KE = E$$

From the definition of work:

$$W = F(\Delta x)$$

$$E = F(\Delta x)$$

Therefore:

$$E = \int_0^x F\,dx$$

From Newton’s original statement of *F=ma*:

$$F = \frac{dm}{dt}v + m\frac{dv}{dt}$$

$$F = \frac{d}{dt}(mv)$$

This, along with the previous equation yields:

$$E = \int_0^x F\,dx$$

We have *dt* and *dx* in the equation.
Let's write it all in terms of *dt*:

$$v = \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

$$dx = v\,dt$$

When the variable changes from *dx* to *dt*,
we must also change the bounds of the integral

$$E = \int_0^x \frac{d}{dt}(mv)\,dx = \int_o^t \frac{d}{dt}(mv)\,v\,dt$$

We can eliminate the *dt*'s so:

$$E = \int_0^t \frac{d}{dt}(mv)v\,dt = \int_0^{mv} v\,d(mv)$$

Notice that the *dt* changed to *d*(*mv*) so the bounds also changed

For velocities approaching *c*, the mass increases

The relativistic mass is: $$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$E = \int_0^{mv} v\,d(mv) = \int_0^{mv} v\,d\left(\frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}\right)$$

Since m_0 is a constant, it can factored out of the integral:

$$E = \int_0^{mv} v\,d\left(\frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}\right) = m_0 \int_0^{v} v\,d\left(\frac{v}{\sqrt{1 - \frac{v^2}{c^2}}}\right)$$

Applying the Quotient Rule:
(keep in mind *v* is a variable and *c* is a constant)

$$d\left(\frac{v}{\sqrt{1 - \frac{v^2}{c^2}}}\right) = \frac{(dv)\sqrt{1 - \frac{v^2}{c^2}} - v\left(\frac{1}{2}\right)\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}\left(-\frac{1}{c^2}\right)(2v\,dv)}{\left(1 - \frac{v^2}{c^2}\right)}$$

Combining these expressions for Energy yields:

$$E = m_0 \int_0^v v\,d\left(\frac{v}{\sqrt{1-\frac{v^2}{c^2}}}\right)$$

$$E = m_0 \int_0^v v\,\frac{(dv)\sqrt{1-\frac{v^2}{c^2}} - v\left(\frac{1}{2}\right)\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}\left(-\frac{1}{c^2}\right)(2v\,dv)}{\left(1-\frac{v^2}{c^2}\right)}$$

$$E = m_0 \int_0^v \left(\frac{v}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{\frac{v^3}{c^2}}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\right)dv$$

To get a common denominator, we will multiply the numerator and denominator of

$$\frac{v}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{by} \quad \frac{(1-\frac{v^2}{c^2})}{(1-\frac{v^2}{c^2})}$$

$$E = m_0 \int_0^v \left(\frac{v}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{\frac{v^3}{c^2}}{(1-\frac{v^2}{c^2})^{\frac{3}{2}}}\right) dv$$

$$E = m_0 \int_0^v \left[\frac{(1-\frac{v^2}{c^2})v}{(1-\frac{v^2}{c^2})^{\frac{3}{2}}} + \frac{\frac{v^3}{c^2}}{(1-\frac{v^2}{c^2})^{\frac{3}{2}}}\right] dv$$

Simplifying:

$$E = m_0 \int_0^v \left[\frac{(1 - \frac{v^2}{c^2})v}{(1 - \frac{v^2}{c^2})^{\frac{3}{2}}} + \frac{\frac{v^3}{c^2}}{(1 - \frac{v^2}{c^2})^{\frac{3}{2}}} \right] dv$$

$$E = m_0 \int_0^v \frac{v - \frac{v^3}{c^2} + \frac{v^3}{c^2}}{(1 - \frac{v^2}{c^2})^{\frac{3}{2}}} dv$$

$$E = m_0 \int_0^v \frac{v}{(1 - \frac{v^2}{c^2})^{\frac{3}{2}}} dv$$

$$E = m_0 \int_0^v \frac{v}{(1 - \frac{v^2}{c^2})^{\frac{3}{2}}} dv$$

$$E = m_0 \int_0^v \frac{v}{(\frac{c^2}{c^2} - \frac{v^2}{c^2})^{\frac{3}{2}}} dv$$

$$E = m_0 \int_0^v \frac{v}{(\frac{c^2 - v^2}{c^2})^{\frac{3}{2}}} dv$$

$$E = m_0 \int_0^v \frac{v}{\frac{(c^2 - v^2)^{\frac{3}{2}}}{c^3}} dv$$

$$E = m_0 \int_0^v \frac{v}{\frac{(c^2 - v^2)^{\frac{3}{2}}}{c^3}} dv$$

$$E = m_0 \int_0^v \frac{c^3 v}{(c^2 - v^2)^{\frac{3}{2}}} dv$$

$$E = m_0 c^3 \int_0^v \frac{v}{(c^2 - v^2)^{\frac{3}{2}}} dv$$

To evaluate the integral we make the substitution:

$$u = c^2 - v^2$$

$$\frac{du}{dv} = -2vdv$$

$$-\frac{1}{2}du = vdv$$

Therefore:

$$E = m_0 c^3 \int_0^v \frac{vdv}{(c^2 - v^2)^{\frac{3}{2}}}$$

$$E = m_0 c^3 \int_{u(v=0)}^{u(v)} \frac{-\frac{1}{2}du}{u^{\frac{3}{2}}}$$

$$E = m_0 c^3 \int_{u(v=0)}^{u(v)} \frac{-\frac{1}{2} du}{u^{\frac{3}{2}}}$$

$$E = -\frac{m_0 c^3}{2} \int_{c^2}^{c^2 - v^2} \frac{du}{u^{\frac{3}{2}}}$$

$$E = -\frac{m_0 c^3}{2} \left(\frac{u^{-\frac{1}{2}}}{-\frac{1}{2}} \Big|_{c^2}^{c^2 - v^2} \right)$$

$$E = m_o c^3 \left(\frac{1}{\sqrt{c^2 - v^2}} - \frac{1}{\sqrt{c^2}} \right)$$

$$E = m_o c^3 \left(\frac{1}{\sqrt{c^2 - v^2}} - \frac{1}{\sqrt{c^2}} \right)$$

$$E = m_o c^3 \left(\frac{1}{\sqrt{c^2 - v^2}} - \frac{1}{c} \right)$$

$$E = m_o c^3 \left(\frac{c}{c\sqrt{c^2 - v^2}} - \frac{\sqrt{c^2 - v^2}}{c\sqrt{c^2 - v^2}} \right)$$

$$E = m_o c^3 \left(\frac{c - \sqrt{c^2 - v^2}}{c\sqrt{c^2 - v^2}} \right)$$

$$E = m_o c^3 \left(\frac{c - \sqrt{c^2 - v^2}}{c\sqrt{c^2 - v^2}} \right)$$

$$E = m_o c^2 \left(\frac{c - \sqrt{c^2 - v^2}}{\sqrt{c^2 - v^2}} \right)$$

$$E = m_o c^2 \left(\frac{c}{\sqrt{c^2 - v^2}} - \frac{\sqrt{c^2 - v^2}}{\sqrt{c^2 - v^2}} \right)$$

$$E = m_o c^2 \left(\frac{c}{\sqrt{c^2 - v^2}} - 1 \right)$$

$$E = m_o c^2 \left(\frac{c}{\sqrt{c^2 - v^2}} - 1 \right)$$

$$E = m_o c^2 \left(\frac{1}{\frac{1}{c}\sqrt{c^2 - v^2}} - 1 \right)$$

$$E = m_o c^2 \left(\frac{1}{\sqrt{(\frac{1}{c^2})(c^2 - v^2)}} - 1 \right)$$

$$E = m_o c^2 \left(\frac{1}{\sqrt{(1 - \frac{v^2}{c^2})}} - 1 \right)$$

$$E = m_o c^2 \left(\frac{1}{\sqrt{(1 - \frac{v^2}{c^2})}} - 1 \right)$$

$$E = c^2 \left(\frac{m_0}{\sqrt{(1 - \frac{v^2}{c^2})}} - m_0 \right)$$

Recall the relativistic mass: $m = \frac{m_0}{\sqrt{(1 - \frac{v^2}{c^2})}}$

Substituting this yields: $E = c^2 (m - m_0)$

or

$$\boxed{E = (m - m_o)c^2}$$

$$E = mc^2$$